PROGRESSIONS

CHAPTER 5

FORM 4

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PRIOR KNOWLEDGE

- Mathematics form 2 chapter 1 – PATTERNS AND SEQUENCE
- Terms that we used
  - PATTERN - are the list of numbers or objects arranged based on a rule or design.
  - NUMBER PATTERN
    - even numbers examples 2, 4, 6, …
    - odd numbers examples 13, 15, 17, …
    - Fibonacci numbers examples 0, 1, 1, 2, 3, …
  - SEQUENCE – is a set of number of objects arranged according to a certain pattern
  - NUMBER SEQUENCE
    - -10, -6, -2, 2, 6, …

      $+4$ $+4$ $+4$ $+4$

      pattern : add 4

      therefore , the set of numbers is a sequence
    - 4, 5, -7, 10, -14, ..

      $+1$ -12 $+17$ -24

      pattern : none

      therefore , the set of numbers is NOT a sequence

TYPES OF PROGRESSIONS

- <u>ARITHMETIC PROGRESSIONS</u>
- <u>GEOMETRIC PROGRESSIONS</u>

ARITHMETIC PROGRESSIONS(AP)

- Is a sequence of numbers such that each term is obtained by adding a constant to the term before it.
- For example
  - 3, 7, 11, 15 …
  - 10, 7, 4, 1, -2 …
- What is the justification to determine that the sequence is an arithmetic progression?
- Common difference is a constant that we add to the previous term.
- Common difference $\longrightarrow$ $d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}$

Identifying AP

Example 1

Determine whether each of the following sequence is an arithmetic progression.

(a) 2, -4, 6, -8, …

(b) $x+y, x-y, x-3y, \ldots$

Solution

$d = T_n - T_{n-1}$

(a)

$$T_2 - T_1 = -4 - 2$$
$$= -6$$
$$T_3 - T_2 = 6 - (-4)$$
$$= 10$$

$10 \neq -6$ Not an AP

(b)

$$T_2 - T_1 = (x-y) - (x+y)$$
$$= x - y - x - y$$
$$= -2y$$
$$T_3 - T_2 = (x-3y) - (x-y)$$
$$= x - 3y - x + y$$
$$= -2y$$

$-2y = -2y$ An AP

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Example 2

Three consecutive terms of an arithmetic progressions are $3m-2$, $m+3$ dan 4.

Find the value of m. [2 *marks*]

Solution

$$d = T_n - T_{n-1}$$

$$(m+3)-(3m-2) = 4-(m+3)$$

$$m = 4$$

The n^{th} term of AP

How to find the term in an arithmetic progressions?

3, 7, 11, 15 …

3 ⟶ T_1 ⟶ a

7 ⟶ T_2 ⟶ 3 + 4 ⟶ $a + d$

11 ⟶ T_3 ⟶ 3 + 4 + 4 ⟶ $a + 2d$

15 ⟶ T_4 ⟶ 3 + 4 + 4 + 4 ⟶ $a + 3d$

T_5 ⟶ $a + 4d$

T_6 ⟶ $a + 5d$

$$T_n = a + (n-1)d$$

$$T_n = a + (n-1)d$$

Where

T_n - is the n^{th} term
n - is the number of term
a - is the first term
d - is the common difference

The n^{th} term of AP

Example 3

Find the 10^{th} term for the arithmetic progression $2,5,8,11,...$

Solution

$a = 2, \quad d = 3, \quad n = 10$

$$T_n = a + (n-1)d$$

$$T_{10} = 2 + (10-1)3$$

$$T_{10} = 29$$

The n^{th} term of AP

Example 4

Find the number of terms for arithmetic progression 0.7, 2.1, 3.5, …, 9.1

Solution

$T_n = a + (n-1)d$

$d = 1.4 \quad a = 0.7 \quad T_n = 9.1$

$$0.7 + (n-1)(1.4) = 9.1$$

$$0.7 + 1.4n - 1.4 = 9.1$$

$$n = 7$$

The n^{th} term of AP

Example 5

The 10th term of an arithmetic progression is 5 and the 18th term is 77. Find the first three terms of this progression.

Solution

$$T_n = a + (n-1)d$$

$$T_{10} = 5$$

$$a + 9d = 5 \quad \text{—— (1)}$$

$$T_{18} = 77$$

$$a + 17d = 77 \quad \text{—— (2)}$$

(2) − (1) $\quad 8d = 72$

$$d = 9$$

$$a + 9d = 5$$

$$a + 9(9) = 5$$

$$a = -76$$

$$T_2 = a + d = -76 + 9 = -67$$

$$T_3 = a + 2d = -76 + 2(9) = -58$$

Therefore the first three terms of this progression is -76, -67, -58

TIPS : to solve arithmetic progression, we usually need to minus the equation

The n^{th} term of AP

Example 6

The first term of an arithmetic progression is 13 and the common difference is 3. Determine

(a) which term is equal to 700,

(b) the first term that exceed 2000.

Solution

$$T_n = a + (n-1)d$$

(a)

$$a = 13,\ d = 3$$

$$T_n = 700$$

$$13 + (n-1)3 = 700$$

$$13 + 3n - 3 = 700$$

$$3n = 690$$

$$n = 230$$

(b)

$$T_n > 2000$$

$$13 + (n-1)3 > 2000$$

$$13 + 3n - 3 > 2000$$

$$3n > 1990$$

$$n > 663\frac{1}{3}$$

$$n = 664$$

THE INVENTION OF ARITHMETIC PROGRESSIONS

Arithmetic Progression was invented by Johann Carl Friedrich Gauss. He is known as the Prince of Mathematics. At the age of 7, his teacher asked the students to sum up from 1 to 100.

$$S_{100} = 1 + 2 + 3 + \dots + 98 + 99 + 100 \quad \text{——} \quad (1)$$

$$S_{100} = 100 + 99 + 98 + \dots + 3 + 2 + 1 \quad \text{——} \quad (2)$$

(1)+(2) : $$2S_{100} = 101 + 101 + 101 + \dots + 101 + 101 + 101$$

100 times 101

$$2S_{100} = 100(101)$$

$$S_{100} = \frac{100}{2}(101)$$

$$S_{100} = \frac{100}{2}(1+100)$$

$$1 = a, 100 = l = T_n$$

$$S_n = \frac{n}{2}(a+l)$$

$$S_n = \frac{n}{2}(a+T_n)$$

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[a+a+(n-1)d] \Rightarrow S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_n = \frac{n}{2}(a+l)$$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Where
S_n - is the sum of the first n^{th} term
n - is the number of term
a - is the first term
d - is the common difference
l – is the last term

SUM OF THE FIRST *n* TERMS, *Sn*, OF AP

Example 7

Given the arithmetic progression -1, 3, 7, …, calculate the sum of the first 50 terms.

Solution

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$a = -1, \quad d = 4, \quad n = 50$$

$$S_{50} = \frac{50}{2}\left[2(-1) + (50-1)4\right]$$

$$S_{50} = 4850$$

SUM OF THE FIRST *n* TERMS, *Sn*, OF AP

Example 8

Given the arithmetic progression 0.7, 0.71, 0.72, …,1.69, calculate the sum of all the terms.

Solution

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}(a+l)$$

$a = 0.7, \quad d = 0.01, \quad l = T_n = 1.69$

$$T_n = 1.69$$

$$0.7 + (n-1)(0.01) = 1.69$$

$$0.7 + 0.01n - 0.01 = 1.69$$

$$0.01n = 1$$

$$n = 100$$

$$S_{100} = \frac{100}{2}(0.7 + 1.69)$$

$$S_{100} = 119.5$$

<u>Alternative method</u>

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{100} = \frac{100}{2}[2(0.7) + (100-1)0.01]$$

$$S_{100} = 119.5$$

SUM OF THE FIRST *n* TERMS, *Sn*, OF AP

Example 9

Find the sum of all the multiples of 8 between 100 and 250.

$$104, 112, 120, \ldots\ldots\ldots, 248$$

Solution

$$a = 104, \; d = 8 \quad l = T_n = 248$$

$$T_n = a + (n-1)d$$

$$T_n = 248$$

$$104 + (n-1)(8) = 248$$

$$104 + 8n - 8 = 248$$

$$8n = 152$$

$$n = 19$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{19} = \frac{19}{2}[2(104) + (19-1)8]$$

$$S_{19} = 3344$$

Alternative method

$$S_n = \frac{n}{2}(a + l)$$

$$S_{19} = \frac{19}{2}(104 + 248)$$

$$S_{19} = 3344$$

SUM OF THE FIRST n TERMS, S_n, OF AP

Example 10

Find the sum from the 10^{th} term to the 20^{th} term of an arithmetic progression 2, 7, 12, 17, …

Solution

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$a = 2, \quad d = 5$

$S_{20} - S_9$

$= \frac{20}{2}[2(2) + (20-1)(5)] - \frac{9}{2}[2(2) + (9-1)(5)]$

$= 990 - 198$

$= 792$

Alternative method

$$T_n = a + (n-1)d$$

$T_{10} = 2 + (10-1)5$

$= 47$ Use T_{10} as new a for a new set of AP

$T_{20} = 2 + (20-1)5$

$= 97$ Use T_{20} as new l for a new set of AP

$$S_n = \frac{n}{2}(a + l)$$

$S_{11} = \frac{11}{2}(47 + 97)$

$= 792$

SUM OF THE FIRST n TERMS, Sn, OF AP

Example 11

The sum of the first five terms for an arithmetic progression is 35 and the sum of the next five terms is 85. Find

(a) first term and common difference,

(b) sum of the first n terms.

Solution

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

(a) $S_5 = 35$

$$S_5 = \frac{5}{2}[2(a) + (5-1)d] = 35$$

$$2a + 4d = 14 \quad \text{—— (1)}$$

$$T_6 + T_7 + T_8 + T_9 + T_{10} = 85$$

$$S_{10} - S_5 = 85$$

$$\left[\frac{10}{2}[2(a) + (10-1)d]\right] - 35 = 85$$

$$2a + 9d = 24 \quad \text{—— (2)}$$

(2) − (1) $\quad 5d = 10$

$$d = 2 \qquad a = 3$$

(b) $S_n = ?$

$$S_n = \frac{n}{2}[2(3) + (n-1)2]$$

$$= \frac{n}{2}(6 + 2n - 2)$$

$$= \frac{n}{2}(4 + 2n)$$

$$S_n = 2n + n^2$$

Obtaining the *n* TERMS using *Sn*

- Example
  - 2, 6, 10, 14, 18, 22, 26, 30
  - These sequence have 8 terms, how to find the seventh term by using sum of the first n terms?

2, 6, 10, 14, 18, 22,	26,	30

T_6 ↑ T_7 ↑ T_8 ↑

$$26 = (2+6+10+14+18+22+26)-(2+6+10+14+18+22)$$

$$T_7 = S_7 - S_6$$

- Another method to find the n term is

$$T_n = S_n - S_{n-1}$$

Obtaining the *n* TERMS using *Sn*

Example 12

Given the sum of the first n terms for an arithmetic progression is $S_n = 2n^2 + 3n$, find the 7^{th} term.

Solution

$$T_n = S_n - S_{n-1}$$

$$S_7 = 2(7)^2 + 3(7)$$
$$= 119$$

$$S_6 = 2(6)^2 + 3(6)$$
$$= 90$$

$$T_7 = S_7 - S_6$$
$$= 119 - 90$$
$$= 29$$

TICKET TO VICTORY

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Example 13

The sum of the first n terms of an arithmetic progressions is given by $S_n = \frac{n}{2}(5n+3)$,

find

(i) the sum of the first *7* terms,

(ii) the 7^{th} term. [4 *marks*]

Solution

(i) $S_n = \frac{n}{2}(5n+3)$

$S_7 = \frac{7}{2}(5(7)+3)$

$S_7 = 133$

$T_n = S_n - S_{n-1}$

(ii) $T_7 = S_7 - S_6$

$S_6 = \frac{6}{2}(5(6)+3) = 99$

$T_7 = 133 - 99$

$T_7 = 34$

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Example 14

It is given that the sum of the first m terms of an arithmetic progression is $S_m = \frac{k+1}{2}(a+7)$, such that k is a constant, a is the first term and 7 is the last term.

(a) Express k in terms of m.

(b) State the range of values of k. [2 *marks*]

Solution

(a) $S_m = \frac{k+1}{2}(a+7)$

$S_n = \frac{n}{2}(a+l)$

Compare

$m = k+1$

$k = m-1$

(b) $S_m = \frac{k+1}{2}(a+7)$

$m = 1,2,3,\ldots$

$k+1 = 1,2,3,\ldots$

$k = 0,1,2,\ldots$

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Example 15

The third term of an arithmetic progression is 4 and the fourth term is 7.

(a) State the common difference of the progression.

(b) Find the sum of the first 25 terms of the progression. [4 *marks*]

$$S_n = \frac{n}{2}(2(a) + (n-1)d$$

Solution

(a) $d = T_4 - T_3$

$= 7 - 4$

$= 3$

(b) $n = 25,\ d = 3,$

$$S_{25} = \frac{25}{2}\left[2(a) + (25-1)3\right]$$

$$T_3 = a + 2d$$

$$4 = a + 2(3)$$

$$a = -2$$

$$S_{25} = \frac{25}{2}\left[2(-2) + (25-1)3\right]$$

$$S_{25} = 850$$

TICKET TO VICTORY
PAPER 1 , NO 14b

Example 16

A string with length of 2030 cm is cut into 28 pieces. The length of each piece follow the arithmetic progression. Starting from the largest string 140 cm, find

(i) the common difference in length of the pieces of string,

[4 *marks*]

Solution

(i) 140, ___, ___, …, ___

28 terms

$$S_n = \frac{n}{2}(2(a) + (n-1)d$$

$$S_{28} = 2030$$

$$\frac{28}{2}\left[2(140) + (28-1)d\right] = 2030$$

$$14(280 + 27d) = 2030$$

$$d = -5$$

TICKET TO VICTORY

PAPER 1, NO 14b

Example 16

A string with length of 2030 cm is cut into 28 pieces. The length of each piece follow the arithmetic progression. Starting from the largest string 140 cm, find

(ii) the sum of the first n terms

(iii) the length of the n^{th} piece.

Solution

(ii) $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_n = \frac{n}{2}[2(140) + (n-1)(-5)]$$

$$= \frac{n}{2}(280 - 5n + 5)$$

$$S_n = \frac{n}{2}(285 - 5n)$$

(iii) $T_n = S_n - S_{n-1}$

$$= \frac{n}{2}(285 - 5n) - \left[\frac{n-1}{2}(285 - 5(n-1)\right]$$

$$= 145 - 5n$$

GEOMETRIC PROGRESSIONS(GP)

- Is a sequence of numbers where each term is obtained by multiplying a constant with the previous term.
- For example
  - -1, 3, -9, 27, ...
  - $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8} \ldots$
- What is the justification to determine that the sequence is a geometric progression?
- Common ratio is a constant that we multiply to the previous term.
- Common ratio $\longrightarrow$ $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots = \frac{T_n}{T_{n-1}}$

Identifying GP

Example 17

Determine whether each of the following sequence is a geometric progression.

(a) 1, 2, 6, 24, …

(b) $p, p^3, p^5, \ldots$

Solution

$$r = \frac{T_n}{T_{n-1}}$$

(a) $\dfrac{T_2}{T_1} = \dfrac{2}{1}$

$= 2$

$\dfrac{T_3}{T_2} = \dfrac{6}{2}$

$= 3$

$2 \neq 3$ Not a GP

(b) $\dfrac{T_2}{T_1} = \dfrac{P^3}{P}$

$= P^2$

$\dfrac{T_3}{T_2} = \dfrac{P^5}{P^3}$

$= P^2$

$P^2 = P^2$ A GP

The n^{th} term of GP

- How to find the term?

5, 15, 45, 135 …

$5 \longrightarrow T_1 \longrightarrow a$

$15 \longrightarrow T_2 \longrightarrow 3 \times 5 \longrightarrow a \times r$

$45 \longrightarrow T_3 \longrightarrow 3 \times 5 \times 5 \longrightarrow a \times r^2$

$135 \longrightarrow T_4 \longrightarrow 3 \times 5 \times 5 \times 5 \longrightarrow a \times r^3$

$T_5 \longrightarrow ar^4$

$T_6 \longrightarrow ar^5$

$$T_n = ar^{n-1}$$

$$T_n = ar^{n-1}$$

Where

T_n - is the n^{th} term

n - is the number of term

a - is the first term

r - is the common ratio

The n^{th} term of GP

Example 18

Find the 6^{th} term for geometric progression $3,-6,12,-24,\ldots$

Solution

$$T_n = ar^{n-1}$$

(a) $a = 3,$ $\quad r = \dfrac{-6}{3}$

$$= -2$$

$$T_6 = 3(-2)^5$$

$$T_6 = -96$$

The n^{th} term of GP

Example 19

Find the number of terms in the geometric progression

(a) 1, 2, 4, 8, …, 512

(b) $16, 8, 4, \ldots, \frac{1}{32}$

Solution

$$T_n = ar^{n-1}$$

(a) $a = 1, \quad r = 2$

$$T_n = 512$$

$$1(2)^{n-1} = 512$$

$$(2)^{n-1} = (2)^9$$

$$n - 1 = 9$$

$$n = 10$$

(b) $a = 16, \; r = \frac{1}{2}$

$$T_n = \frac{1}{32}$$

$$16\left(\frac{1}{2}\right)^{n-1} = \frac{1}{32}$$

$$(2)^4\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^5}$$

$$(2)^4(2)^{-1(n-1)} = 2^{-5}$$

$$2^{4-n+1} = 2^{-5}$$

$$4 - n + 1 = -5$$

$$n = 10$$

TIPS : to make it easier finding the index, please memorise some of its value

The n^{th} term of GP

Example 19

Given the first three terms in the geometric progression are $x-4, x, 5x-12$ where all the terms are positive. Find

(a) the value of x (b) the 7^{th} term

Solution

(a) $\frac{x}{x-4}=\frac{5x-12}{x}$ $r=\frac{T_n}{T_{n-1}}$

$$(x-4)\times\left(\frac{x}{x-4}\right)=\left(\frac{5x-12}{x}\right)\times(x-4)$$

$$x=\left(\frac{5x-12}{x}\right)\times(x-4)$$

$$x\times x=\left(\frac{5x-12}{x}\right)\times(x-4)\times x$$

$$x^2=(5x-12)\times(x-4)$$

$$x^2=5x^2-32x+48$$

$$4x^2-32x+48=0$$

$$x^2-8x+12=0$$

$$(x-2)(x-6)=0$$

$x=2$ or $x=6$

$x\neq 2$ $x=6$

(b) $T_7=?$ $T_n=ar^{n-1}$

$$r=\frac{6}{6-4}$$

$$=3$$

$$a=6-4$$

$$=2$$

$$T_7=2(3)^{7-1}$$

$$T_7=1458$$

The n^{th} term of GP

Example 20

In a geometric progression, the third term exceeds the first term by 9 and the sum of the second term and the third term is 18. Find the first term and the common ratio.

Solution

$$T_n = ar^{n-1}$$

$$T_3 - T_1 = 9$$

$$T_2 + T_3 = 18$$

$$ar^2 - a = 9 \quad \text{——— (1)}$$

$$ar + ar^2 = 18 \quad \text{——— (2)}$$

(2) ÷ (1)

$$\frac{ar + ar^2 = 18}{ar^2 - a = 9}$$

$$\frac{ar(1+r) = 18}{a(r^2-1) = 9}$$

$$\frac{ar(1+r) = 18}{a(r-1)(r+1) = 9}$$

$$\frac{r}{r-1} = 2$$

$$r = 2$$

$$a(2)^2 - a = 9$$

$$a = 3$$

TIPS : to solve geometric progression, we usually need to divide the equation

The n^{th} term of GP

Example 21

Find the smallest terms in the geometric progression 16, 24, 36, …that exceeds 150

Solution

$$T_n = ar^{n-1}$$

$$a = 16, \quad r = \frac{24}{16} = 1.5, \quad T_n > 150$$

$$ar^{n-1} > 150$$

$$16(1.5)^{n-1} > 150$$

$$(1.5)^{n-1} > 9.375$$

$$\log_{10}(1.5)^{n-1} > \log_{10} 9.375$$

$$(n-1)\log_{10} 1.5 > \log_{10} 9.375$$

$$(n-1)\frac{\log_{10} 1.5}{\log_{10} 1.5} > \frac{\log_{10} 9.375}{\log_{10} 1.5}$$

$$n-1 > \frac{\log_{10} 9.375}{\log_{10} 1.5}$$

$$n-1 > 5.520$$

$$n > 6.520$$

$$n = 7$$

TIPS : when it comes to find the value with decimal or big numbers, use log

SUM OF THE FIRST *n* TERMS, *Sn*, OF THE GEOMETRIC PROGRESSION

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \quad \text{———} \quad (1)$$

(1) × r : $$rS_n = ar + ar^2 + ar^3 + ar^4 \ldots + ar^{n-1} + ar^n \quad \text{———} \quad (2)$$

(1) – (2) : $$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

$$- \quad rS_n = ar + ar^2 + ar^3 + ar^4 \ldots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}, r \neq 1$$

commonly used when $r < 1$

SUM OF THE FIRST *n* TERMS, *Sn*, OF THE GEOMETRIC PROGRESSION

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \quad \text{———} \quad (1)$$

(1) × r : $$rS_n = ar + ar^2 + ar^3 + ar^4 \ldots + ar^{n-1} + ar^n \quad \text{———} \quad (2)$$

(2) – (1) : $$rS_n = ar + ar^2 + ar^3 + ar^4 \ldots + ar^{n-1} + ar^n$$

$$- \quad S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

$$rS_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n - 1)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

commonly used when $r > 1$

$$S_n = \frac{a(r^n - 1)}{r - 1}, r > 1$$

$$S_n = \frac{a(1 - r^n)}{1 - r}, r < 1$$

Where
S_n - is the sum of the first n^{th} term
n - is the number of term
a - is the first term
r - is the common ratio

SUM OF THE FIRST *n* TERMS, *Sn*, OF GP

Example 22

Given the geometric progression 8, -24, 72, …, calculate the sum of the first 6 terms.

Solution

$$S_n = \frac{a(r^n - 1)}{r - 1}, r > 1$$

$$S_n = \frac{a(1 - r^n)}{1 - r}, r < 1$$

$$r = \frac{-24}{8} = -3$$

$$a = 8, \quad r = -3, \quad n = 6$$

$$S_6 = \frac{8\left(1 - (-3)^6\right)}{\left(1 - (-3)\right)}$$

$$= \frac{8(1 - 729)}{1 + 3}$$

$$= \frac{8(-728)}{4}$$

$$S_6 = -1456$$

SUM OF THE FIRST *n* TERMS, *Sn*, OF GP

Example 23

Given the geometric progression $\frac{5}{4}, \frac{5}{2}, 5, \ldots, 160$, calculate the sum of all the terms.

Solution

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}, r > 1$$

$$S_n = \frac{a(1 - r^n)}{1 - r}, r < 1$$

$$a = \frac{5}{4}, \quad r = 2, \quad T_n = 160$$

$$\frac{5}{4}(2)^{n-1} = 160$$

$$2^{n-1} = 128$$

$$2^{n-1} = 2^7$$

$$n - 1 = 7$$

$$n = 8$$

$$S_8 = \frac{\frac{5}{4}(2^8 - 1)}{2 - 1}$$

$$S_8 = \frac{1275}{4}$$

$$= 318\frac{3}{4}$$

Example 24

Amin save RM1 on the first day, RM2 on the second day, RM4 on the third day and so on. How many days that he has to save to get the sum of RM4 095?

Solution

$$S_n = \frac{a(r^n - 1)}{r - 1}, r > 1$$

$$S_n = \frac{a(1 - r^n)}{1 - r}, r < 1$$

$$a = 1, \quad T_2 = 2, \; T_3 = 4$$

$$S_n = 4095, \qquad r = 2 > 1$$

$$\frac{1(2^n - 1)}{2 - 1} = 4095$$

$$2^n = 4096$$

$$\log_{10} 2^n = \log_{10} 4096$$

$$n \log_{10} 2 = \log_{10} 4096$$

$$(n)\frac{\log_{10} 2}{\log_{10} 2} = \frac{\log_{10} 4096}{\log_{10} 2}$$

$$n = \frac{\log_{10} 4096}{\log_{10} 2}$$

$$n = 12$$

DETERMINE THE SUM TO INFINITY OF GEOMETRIC PROGRESSION

Look at this sequence $4,2,1,\frac{1}{2},\frac{1}{4},\frac{1}{16},\frac{1}{32},\ldots$

Find sum to infinity for this sequence

n	r^n	S_n
1	$\frac{1}{2}=0.5$	4
2	$\left(\frac{1}{2}\right)^2=\frac{1}{4}=0.25$	6
3	$\left(\frac{1}{2}\right)^3=\frac{1}{8}=0.125$	7
4	$\left(\frac{1}{2}\right)^4=\frac{1}{16}=0.0625$	7.5
5	$\left(\frac{1}{2}\right)^5=\frac{1}{32}=0.03125$	7.75
10	$\left(\frac{1}{2}\right)^{10}=\frac{1}{1024}=0.000976$	7.992…
20	$\left(\frac{1}{2}\right)^{20}=\frac{1}{1048576}=0.00000095$	7.999…

$a=4, \quad r=\frac{1}{2}$

$$S_n=\frac{a(1-r^n)}{1-r}, r<1$$

$$S_\infty=\frac{a}{1-r}$$

where $|r|<1$ or $-1<r<1$

SUMMARY FOR THE SUM FORMULAE IN GEOMETRIC PROGRESSION

Range of r	Formula
$r > 1$	$S_n = \frac{a(r^n - 1)}{r - 1}$
$r < 1$	$S_n = \frac{a(1 - r^n)}{1 - r}$
$-1 < r < 1$	$S_\infty = \frac{a}{1 - r}$

SUM TO INFINITY OF GP

Example 25

Determine whether the following geometric progression has the sum to infinity. Hence, find the sum to infinity.

(a) 2, -8, 32, -108, …

(b) $4, 1, \frac{1}{4}, \frac{1}{16}, \ldots$

Solution

$$S_\infty = \frac{a}{1-r}$$

(a) $r = \frac{T_2}{T_1} = \frac{-8}{2} = -4$

$r < -1, \quad S_\infty$ not exist

(b) $r = \frac{T_2}{T_1} = \frac{1}{4}$

$-1 < r < 1, \quad S_\infty$ exist

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{4}{1-\frac{1}{4}}$$

$$S_\infty = 5\frac{1}{3}$$

SUM TO INFINITY OF GP

Example 26

Given that the first term of geometric progression is 8 and common ratio r. A second geometric progression has first term 10 and common ratio $\frac{1}{4}r$. The two progressions have the same sum to infinity, S_∞ . Find the value of r and the value of S_∞ .

Solution

$$S_\infty = \frac{a}{1-r}$$

$$a = 8, \quad r = r$$

$$a = 10, \quad r = \frac{1}{4}r$$

$$\frac{8}{1-r} = \frac{10}{1-\frac{1}{4}r}$$

$$8\left(1-\frac{1}{4}r\right) = 10(1-r)$$

$$8 - 2r = 10 - 10r$$

$$8r = 2$$

$$r = \frac{1}{4}$$

$$S_\infty = \frac{8}{1-\frac{1}{4}}$$

$$S_\infty = 10\frac{2}{3}$$

SUM TO INFINITY OF GP

We can find the repeating decimal value for fractions. For examples :

(a) $\frac{1}{3} = 0.3333...$

(b) $\frac{5}{7} = 0.714285714285...$

Can we find fractions from the repeating decimal?

Example 27

Express the repeating decimals 0.232323… in the form of sum to infinity of geometric progressions.

Solution

$$0.23 + 0.0023 + 0.000023 + 0.00000023 + ...$$

$a = 0.23,\quad r = \frac{0.0023}{0.23} = 0.01$

$$S_\infty = \frac{0.23}{1-0.01}$$

$$S_\infty = \frac{23}{99}$$

$$S_\infty = \frac{a}{1-r}$$

Example 28

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Three consecutive terms of a geometric progression are 32, p, q. It is given that the sum of these three terms is 26.

Find the possible values of p and of q. [3 *marks*]

Solution

$$r = \frac{T_n}{T_{n-1}}$$

$$32 + p + q = 26$$

$$\frac{p}{32} = \frac{q}{p}$$

$$p^2 = 32q$$

$$q = \frac{p^2}{32}$$

$$32 + p + \frac{p^2}{32} = 26$$

$$1024 + 32p + p^2 = 832$$

$$p^2 + 32p + 192 = 0$$

$$(p+8)(p+24) = 0$$

$$p = -8, -24$$

$$q = 2, 18$$

Example 29

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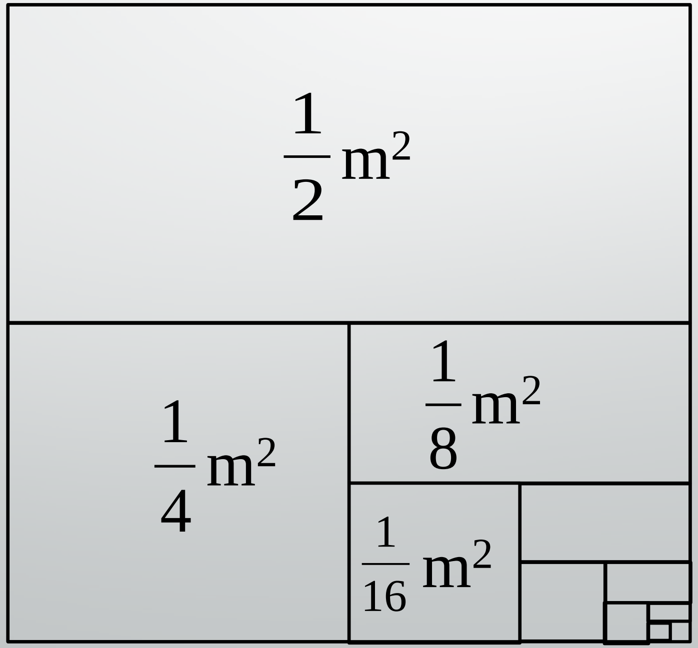
Diagram shows a square sheet with the sides of 1 meter. The sheet is folded into two equal rectangles. The rectangle is then folded into two equal squares and the process continues indefinitely. The shapes formed by the process become a series of rectangles and squares with the area, in m^2, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$



Example 29

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(a) Determine the first rectangle or square that has an area of less than 0.0001 m^2.
State whether it is a rectangle or a square. [3 *marks*]

Solution

$T_n = ar^{n-1}$

(a) $T_n = ar^{n-1} < 0.0001$

$a = \frac{1}{2}, \quad T_2 = \frac{1}{4}, \quad r = \frac{1}{2}$

$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} < 0.0001$

$\left(\frac{1}{2}\right)^{n-1} < 0.0002$

$(n-1)\log_{10}\frac{1}{2} < \log_{10} 0.0002$

$(n-1)\log_{10} 0.5 < \log_{10} 0.0002$

$(n-1)\frac{\log_{10} 0.5}{\log_{10} 0.5} > \frac{\log_{10} 0.0002}{\log_{10} 0.5}$

$n-1 > \frac{\log_{10} 0.0002}{\log_{10} 0.5}$

$n-1 > 12.29$

$n > 13.29$

$n = 14$

TIPS : the value for log <1 is negative

Series of rectangle

$\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots \quad T_1, T_3, T_5, \ldots$

Series of square

$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots \quad T_2, T_4, T_6, \ldots$

$n = 14$ Square

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Example 29

(b) Find the total area, in m^2, of the rectangle in this series. [3 *marks*]

Solution

List of area of rectangle $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$

$$r = \frac{1}{4}, \quad -1 < r < 1, \quad S_\infty \text{ exist}$$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{4}}$$

$$S_\infty = \frac{2}{3}$$

Obtaining the *n* TERMS using *Sn*

Example 30

Given the sum of the first n terms for a geometric progression is $S_n = 1 - 4^n$, find the 7^{th} term.

Solution

$$T_n = S_n - S_{n-1}$$

$$S_7 = 1 - 4^7$$
$$= 1 - 16384$$
$$= -16383$$

$$S_6 = 1 - 4^6$$
$$= 1 - 4096$$
$$= -4095$$

$$T_7 = S_7 - S_6$$
$$= -16383 - (-4095)$$
$$= -16383 + 4095$$
$$= -12288$$

LIST OF THE FORMULAE IN PROGRESSION THAT IS PROVIDED IN SPM.

$$T_n = a + (n-1)d$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{n}{2}(2(a) + (n-1)d$$

$$S_n = \frac{a(r^n - 1)}{r-1} = \frac{a\left(1 - r^n\right)}{1-r}, r \neq 1$$